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Dynamic stability of running: The effects of speed and leg amputations on the maximal Lyapunov exponent

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In this paper, we study dynamic stability during running, focusing on the effects of speed, and the use of a leg prosthesis. We compute and compare the maximal Lyapunov exponents of kinematic time-series data from subjects with and without unilateral transtibial amputations running at a wide range of speeds. We find that the dynamics of the affected leg with the running-specific prosthesis are less stable than the dynamics of the unaffected leg and also less stable than the biological legs of the non-amputee runners. Surprisingly, we find that the center-of-mass dynamics of runners with two intact biological legs are slightly less stable than those of runners with amputations. Our results suggest that while leg asymmetries may be associated with instability, runners may compensate for this effect by increased control of their center-of-mass dynamics. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4837095>]

In order to understand the combined effects of speed, stability, and the use of leg prostheses, it is important to explore the dynamical details of running. Nonlinear time-series analysis of kinematic gait data can elucidate these details. There have been a number of experimental studies of the dynamics of running. To our knowledge, however, no one has explored the dynamic stability of runners with leg amputations, a population to whom dynamical stability seems an especially important issue. Using nonlinear time-series analysis on motion-capture data from treadmill studies, we analyzed the gait dynamics of runners with and without a unilateral transtibial amputation, from a slow run up to each individual's top speed. We used standard delay-coordinate embedding techniques to reconstruct the dynamics from scalar time-series traces of the positions of various anatomical markers (e.g., the height of the sacrum or the sagittal-plane angle of the right knee), then we calculated the maximal Lyapunov exponent λ_1 of each resulting trajectory. We found that stability decreased at faster speeds for all runners, with or without amputations—although not monotonically and with one exception: the vertical dynamics of the center of mass (COM) in runners with amputations. We also found that the lower-limb dynamics were less stable (viz., higher λ_1) for the affected leg of runners with an amputation than for their unaffected leg—and less stable than either leg of the non-amputee runners. All lower-limb λ_1 values increased with running speed, but the inter-leg and inter-group relationships remained largely the same. Surprisingly, the results showed that the center-of-mass dynamics of non-amputee runners were generally less stable than for runners with an amputation. This suggests that asymmetries may be lead to instability in the leg dynamics that is compensated for by increased control of the center of mass.

I. INTRODUCTION

Analysis of locomotion dynamics elucidates temporal variations in gait patterns and also has the potential to lead to a better understanding of stability. Nonlinear time-series analysis techniques have been used to study various aspects of human walking, including differences between normal and pathological walking gait (e.g., Refs. 9 and 18), the effects of age and illness,^{6,29} synchronization when two people walk side-by-side,²⁸ recognition of an individual from his or her gait,¹³ and stability of walking in the face of continuous perturbations.²⁵ The goal of our study was to explore the effects of speed, stability, and leg prosthesis use in the dynamics of running. At moderate speeds, a runner can be modelled as a bouncing spring-mass system, whereas walking can be represented as a series of inverted-pendulum arcs. A number of interesting models of the dynamics of running have been developed in the biomechanics, robotics, and nonlinear dynamics communities (e.g., Ref. 19), some of which were specifically constructed to explore stability issues.⁷ Only a few studies involved nonlinear analysis of laboratory data from human runners (e.g., Refs. 20 and 26) and few of these explored the temporal details of the dynamics. Further, the effects of prosthesis use on these dynamics have not, to our knowledge, been studied at all. (A recent paper³⁴ addresses the effects of prosthesis use on walking, but walking and running have distinctly different dynamics, and running-specific prostheses are mechanically different than walking prostheses.)

To explore these dynamics, we collected data from 17 subjects running on an instrumented treadmill across a wide range of speeds (3–9 m/s). Six of these subjects had a unilateral transtibial amputation and eleven had two intact biological legs. The time-series data included the xyz positions of

reflective markers placed on the body, gathered via motion-capture cameras over a number of gait cycles. We reconstructed the center-of-mass dynamics using delay-coordinate embedding on various scalar projections of these raw data. We reconstructed the limb dynamics by converting the 3D positions to joint angles and then embedded those angle traces. Finally, we calculated the maximal Lyapunov exponent λ_1 of each of the embedded trajectories using the algorithm of Kantz.²¹

Quantifying the dynamic stability of human locomotion is not a trivial task. Early work used λ_1 as a proxy for stability in walking studies,⁹ an approach that has been employed extensively since then (e.g., Refs. 4, 5, 24, and 25). Later authors unpacked the notion of stability in more detail. Full *et al.*, for instance, suggested a detailed approach that decomposes each trajectory into limit cycles and quantifies the rates of recovery from perturbations in different state-space directions.¹⁵ In a recent review paper,³ Bruijn *et al.* list three requirements for gait stability: recovery from small perturbations, recovery from large perturbations, and a definition of “large” that encompasses all foreseeable perturbations. They found that the maximal Lyapunov exponent is not only a good measure for quantifying the ability to recover from small perturbation but also a good predictive measure of fall risk.

Unlike Ref. 3, we focus neither on fall risk nor on walking. Rather, we are interested in the small-scale perturbations that occur during normal running (e.g., irregularities in the running surface) and how they evolve over time scales smaller than the stride interval. We take the straightforward dynamical systems interpretation of the term “stability”—that is, whether those perturbations grow or shrink. We quantify the answer to that question using the maximal Lyapunov exponent of the dynamics, calculated following the classic dynamical systems approach. An important issue that underlies all of this analysis is the stationarity of the dynamics: e.g., whether the system reacts differently to endogenous and exogenous perturbations. All of these issues are discussed later in this paper.

A nonlinear time-series analysis approach is useful not only for exploring the nonlinear dynamics of running but also for assessing the sensitivity of those dynamics to perturbations. A better understanding of these effects could potentially inform the design of better prostheses for running. A careful assessment of dynamics is also useful for understanding the intertwined roles of symmetry and stability. Seeley *et al.*³⁰ and Gundersen *et al.*,¹⁷ for instance, demonstrated that healthy walking gait is bilaterally symmetrical, even though slight asymmetries may develop to accommodate for changing environmental factors. Skinner and Effney³¹ found significant bilateral asymmetries in the lower-limb kinematics of people with leg amputations during walking; Enoka *et al.*¹² found similar asymmetries in running. During running and sprinting, Grabowski *et al.* determined that people with a unilateral transtibial amputation applied significantly less force to the ground with their affected leg than their unaffected leg.¹⁶ It is not known, however, if that kind of force asymmetry affects the dynamic stability of gait. Variability and asymmetry are not necessarily detrimental;

in the introduction to the 2009 focus issue of CHAOS on “Bipedal Locomotion—From Robots to Humans,” Milton²⁷ writes, “Thus it is possible that a certain amount of kinematic variability in certain aspects of performance might be indicative of a healthier dynamical system.” A comparison of the gait dynamics of non-amputee runners to those of runners with a unilateral transtibial amputation may elucidate these subtle effects.

The research reported in this paper was driven by three hypotheses:

1. For individuals with or without a transtibial amputation, dynamic stability will decrease at faster running speeds.

This hypothesis was based on the work of England and Granata, who found that faster walking speeds lead to larger λ_1 (viz., less stability).¹¹ We expected a similar relationship between speed and stability during running.

2. The λ_1 of the lower-limb dynamics of runners with a unilateral transtibial amputation will be asymmetric, across all speeds.

This followed from the geometric asymmetry of the dynamical system, defined as the notable anthropomorphic differences (mass and moment of inertia) between the affected and unaffected legs, as well as the loss of muscular control in the affected leg.

3. The λ_1 of the center-of-mass dynamics of runners with a unilateral transtibial amputation will be greater than in non-amputee runners.

This hypothesis was based on the rationale that symmetry in the lower limbs poses a challenge to maintaining overall stability during locomotion.

Our study confirmed our first two hypotheses. All but one of the λ_1 values increased with running speed. The exception was the dynamics reconstructed from the vertical position of the center of mass in runners with amputations, where the λ_1 first rose and then fell as running speed increased. We found that lower-limb dynamics were generally less stable (viz., higher λ_1) for the affected leg of runners with amputations than for their unaffected leg—or than for either leg of the non-amputee runners. All lower-limb λ_1 values increased with running speed, but the inter-leg and inter-group relationships remained largely the same. Surprisingly, though, our results showed that the center-of-mass dynamics of non-amputee runners were slightly less stable than in runners with a unilateral transtibial amputation.

Sections II and III describe how the data for this study were collected and analyzed. The results are presented in Sec. IV and discussed in Sec. V.

II. DATA COLLECTION

A total of 17 subjects—6 runners (4 male and 2 female) with a unilateral transtibial amputation and 11 runners (8 male and 3 female) without amputations—participated in the study described in this paper. In the rest of this document, members of these two groups are designated with the acronyms WA and NA, respectively. All of the experiments occurred at the Biomechanics Laboratory of the Orthopedic Specialty Hospital (Murray, Utah). A photograph of the

setup is shown in Figure 1. All subjects gave informed written consent according to the Intermountain Healthcare IRB approved protocol. Each WA subject used his or her own sprint-specific passive-elastic prosthesis during the tests. We measured each subject's height, mass, prosthesis mass, and standing leg lengths. We defined leg length as the vertical distance from the greater trochanter to the floor during standing. We measured the length of the affected leg of each WA subject when it was unloaded, by having the individual stand with a 2 in. wooden block under the unaffected leg.

Subjects performed a series of constant-speed running trials on a custom high-speed treadmill (Treadmetrix, Park City, UT). Each trial consisted of at least ten strides except for top-speed trials, which consisted of 8 strides. After a brief warm up, subjects started the series of running trials at 3 m/s. Each subsequent trial speed was incremented by 1 m/s until subjects reported that they were approaching their top speed. Smaller speed increments were then employed until subjects reached their top speed, defined as the speed at which they could not maintain their position on the treadmill for more than eight strides.³³ Subjects were allowed as much time as desired between trials for full recovery. The pelvis position was defined by reflective markers attached to the anterior and posterior iliac spines and iliac crests of the right and left sides. We used reflective marker clusters to define the thigh and shank segments. In order to define the hip and knee joint centers, we also placed reflective markers on the greater trochanters and the medial and lateral femoral condyles of the right and left legs. We used a marker placed over the sacrum as a proxy for the center of mass. We used motion-capture cameras (Motion Analysis Corporation, Santa Rosa, CA) to measure the 3D positions of those markers at a rate of 300 frames per second, then calculated the joint angles from those data using Visual3D software (C-Motion, Inc., Germantown, MD). We did not normalize the timebase of each data set to the average stride period, as is done in some gait studies, because that operation would obscure the speed effects in which we were interested.³⁵

We used the spatial position of the sacrum marker at the base of the spine to study the center-of-mass dynamics. This is, of course, an approximation. The sacrum location is close



FIG. 1. Subject with a unilateral transtibial amputation running on a high-speed instrumented treadmill.

to the overall body COM when a person stands in the standard anatomical position. However, when a person runs, their COM position moves within the body. It is possible to estimate the COM location using a segmental approach, but this methodology relies on many assumptions and estimates about human body segment parameters. Furthermore, there are no established methodologies for estimating how an amputation and/or the use of a running-specific prosthesis affects the position or movement of the COM. Thus, we chose to use the sacrum marker as a proxy for estimating the COM location and studying its dynamics during running.

III. TIME-SERIES ANALYSIS

The time-series data described in Sec. II comprised time-series traces of dozens of joint positions in 3D space. To reconstruct the locomotion dynamics from these data, we used delay-coordinate embedding. Provided that the underlying dynamics and the observation function h that produces the measurement $x(t)$ from the underlying state variables X of the dynamical system are both smooth and generic, the delay-coordinate map

$$F(\tau, m)(x) = ([x(t) \ x(t + \tau) \ \cdots \ x(t + (m)\tau)]), \quad (1)$$

with delay τ from a d -dimensional smooth compact manifold M to R^m is a diffeomorphism on M if the embedding dimension m is greater than $2d$.³² Here, M is the dynamics of the human body; h is the measurement executed by the motion-capture system, plus the post-processing involved in the conversion from marker positions to joint angles.

Since the body is a coupled dynamical system, one should theoretically be able to use delay-coordinate embedding to reconstruct its d -dimensional dynamics from any single joint position (or angle). Here, though, we wished to focus on smaller units of the body. To this end, we used the medio-lateral (x), anterior-posterior (y) and vertical (z) position coordinates of the sacrum marker to assess the center-of-mass dynamics. To explore the lower-limb dynamics, we used the sagittal plane knee- and hip-joint angles. Examples of these data can be seen in Figure 2, which shows traces of the knee-angle data from two of the runners in this study, one NA and one WA subject. The temporal patterns in the left- and right-knee angles of the NA runner are very similar, though they are of course roughly 180 degrees out of phase. There is an obvious difference, however, between the knee angles of the affected and unaffected legs of the WA subject. All four traces—both knees of both runners—demonstrate largely, but not completely, periodic motion.

To reconstruct the state-space dynamics from these time-series data, we followed standard procedures regarding the choice of appropriate values for the embedding parameters: the minimum of the mutual-information curve¹⁴ as an estimate of the delay τ and the false-neighbor technique of Ref. 23, with a threshold of 10%, to estimate the embedding dimension m . Figure 3 shows the mutual information and false-neighbor curves for the time-series data of Figure 2. To perform these calculations, we used TISEAN's `mutual` and `false_nearest` tools.¹ The

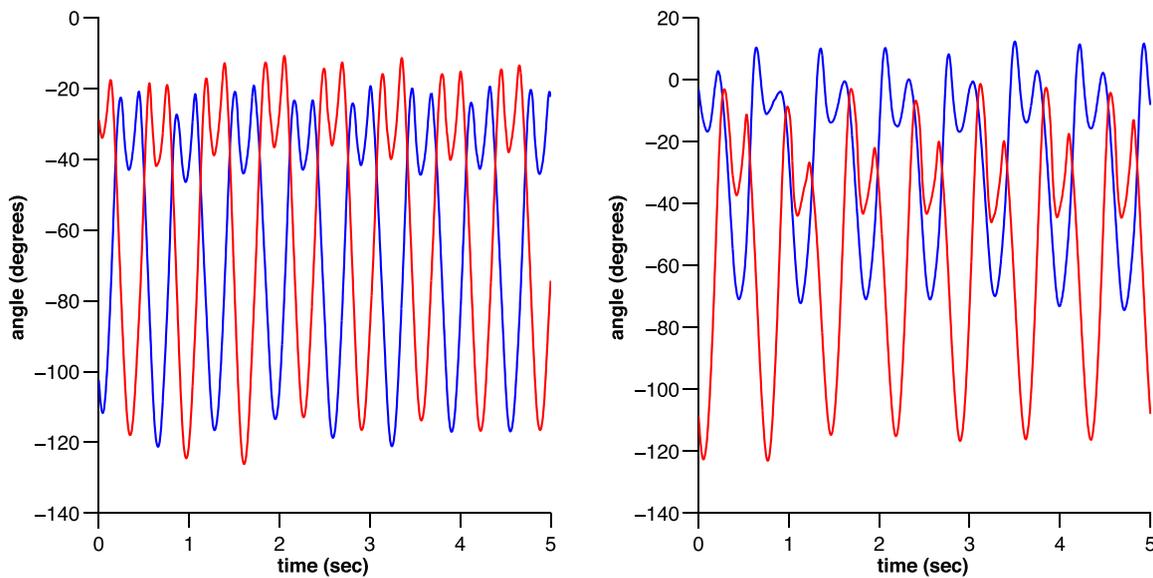


FIG. 2. Sagittal-plane knee angles for (left) a non-amputee runner and (right) a runner with a unilateral transtibial amputation, both running at 4 m/s. Blue and red correspond to the left and right leg, respectively, of the NA runner, and to the affected and unaffected leg, respectively, of the WA subject. 0° is full extension; negative angles correspond to flexion of the joint.

corresponding embeddings are shown in Figure 4. For both legs of both subjects, $m=3$ was sufficient to unfold the dynamics and the first minima of the mutual information curves occurred between $52\Delta n$ and $56\Delta n$, where the sampling interval Δn is $1/300$ th of a second. The embedded trajectories have a characteristic figure-eight shape that reflects the general pattern of running gait, but with visible stride-to-stride variations.

To study these patterns and variations, we employed the algorithm of Kantz,²¹ as instantiated in TISEAN's `lyap_k` tool, to estimate the maximal Lyapunov exponent λ_1 of the embedded data. First, we plotted the log of the expansion rate $S(\Delta n)$ versus time Δn and verified that the curves were of the appropriate shape: a scaling region followed by a horizontal asymptote, which should occur when the time horizon of the algorithm is large enough to allow neighboring trajectories to stretch across the diameter of the attractor. We then fit a line to that scaling region and determined its slope. If the R^2 value of that fit was less than 0.9, we discarded that trial. Note that this approach differs from some of the λ_1 -based work in the biomechanics literature. Ref. 9, for instance, defines a short-term λ_S as the slope calculated using the heights of the $S(\Delta n)$ versus time Δn curve at two fixed landmarks (0 and 1.0 strides) and a long-term λ_L that is calculated over a fixed multiple-stride interval (usually the height of the curve at 5 and 10 strides). Different authors use different a priori landmarks for these calculations; Ref. 4, for instance, uses the points at 0 and 0.5 strides to calculate λ_S and the points at 4 & 10 strides to calculate λ_L . We chose to follow the standard practice in the nonlinear dynamics literature, choosing the segment of the $S(\Delta n)$ vs. Δn curve to fit from its actual geometry, not some a priori fixed interval. Figure 5 shows the $\log S(\Delta n)$ versus Δn curves for the time-series data in Figure 2, embedded using the τ and m values suggested by the curves in Figure 3. These results indicated that the dynamics of both knees of each of these two

runners was sensitively dependent on initial conditions, with λ_1 ranging from 0.0928 to 0.1220. In both subjects, the λ_1 values differed between the two legs, but the difference was more pronounced for the WA subject. This pattern is discussed at more length in Sec. IV.

Several different timescales are involved in this analysis. All λ_1 values that are reported in this paper are in units of the inverse of the sampling interval of the data (3.3 ms). To convert these λ_1 values to inverse seconds, as in the previous paragraph, one multiplies them by the sampling frequency (300). The λ_1 value of 0.0955 for Figure 5(a), for instance, indicates that the e-folding time of the dynamics is 10.5 sample intervals—or 0.035 s. The stride interval of the runner is another important timescale, and it varies across subjects and speeds. In many gait dynamics studies, the time-base of each trace is normalized to the average stride or step period in that trial. That makes sense when one is studying the long-term dynamics across multiple strides. Our work, however, focuses on the small-scale perturbations that occur during normal running and how those perturbations evolve over time scales that are smaller than the stride interval. All of our $S(\Delta n)$ vs. Δn curves, like Figure 5, saturated after 10–20 sample intervals—i.e., 0.033–0.067 s. These timescales are far shorter than the stride intervals in this study, which ranged from 0.44 to 0.77 s. The stride interval of the NA subject in Figure 2, for instance, was 0.65 s, which is 18.7 times the e-folding time (0.035 s) of the associated dynamics that is indicated by Figure 5(a). If the stride interval were shorter, the same e-folding time would be a larger portion of that interval. Normalizing the timebases of each trace to stride interval, then, make the same absolute e-folding time look smaller as the stride rate increases with increasing running speed—simply because the “time” units in which it is measured are getting larger. This effect would artificially lower the value of λ_1 with increasing running speed, so we do not take that approach here.

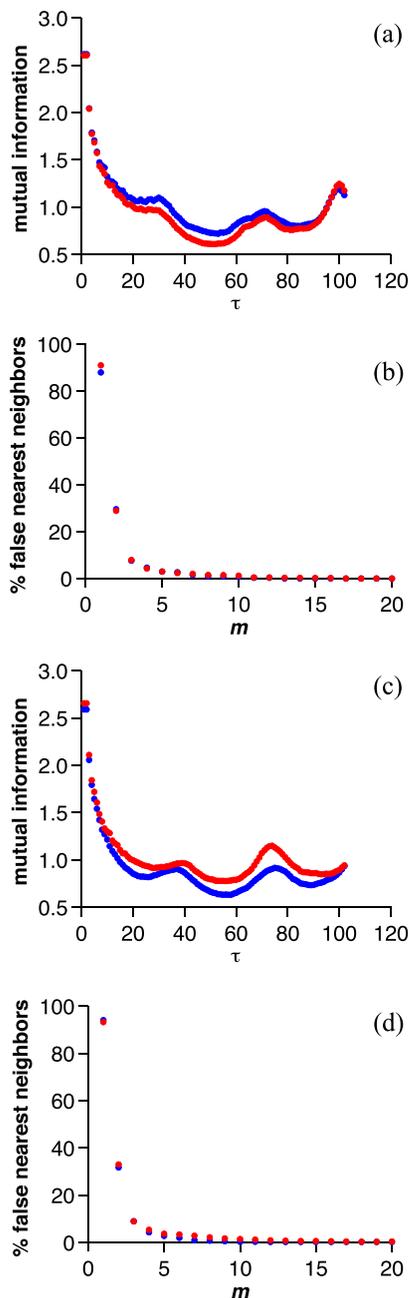


FIG. 3. Estimating embedding parameters for the data of Figure 2: mutual information as a function of the delay τ , plotted in units of the sampling rate $\Delta n = 1/300$ th of a second, and % false near neighbors as a function of the dimension m . The minima of the mutual information curves occur near $\tau = 52\Delta n$ (i.e., 173 ms) for both knees of the non-amputee (“NA”) runner and $\tau = 55\Delta n$ (i.e., 183 ms) for both knees of the runner with a unilateral transtibial amputation (the “WA” subject). All four false near neighbor curves drop to 10% at $m=3$. Color code as in the previous figure: blue and red correspond to the left and right leg, respectively, of the NA runner, and to the affected and unaffected leg, respectively, of the WA subject.

All of the nonlinear time-series analysis algorithms mentioned in this section are known to be sensitive to data and parameter effects.²² These systematic uncertainties preclude the use of traditional statistics to assess or compare their results, but there are other ways to perform “due diligence.” We validated all λ_1 calculations by repeating them for a range of values of the various algorithmic parameters. Of these, the two most critical are the embedding

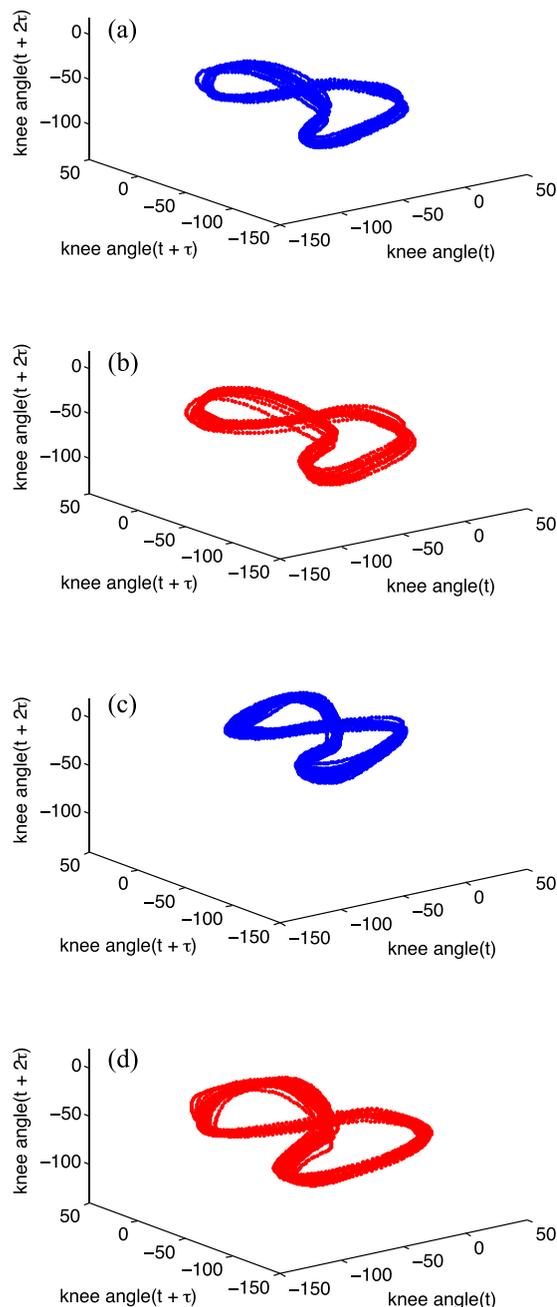


FIG. 4. Delay-coordinate embeddings of the traces in Figure 2 with the τ and m values suggested by the curves in Figure 3. Again, time (t) is in units of Δn , the inverse of the 300 Hz sampling rate of the time series.

dimension m and the scale parameter (ϵ) in the `lyap_k` algorithm.³⁶ We discarded any trial for which this process produced inconsistent results (i.e., large variation with m and/or ϵ). As mentioned above, we also discarded all results from $S(\Delta n)$ versus Δn curves that did not have a clear scaling region. We repeated all of these calculations on seven traces (right and left knee- and hip-joint angles, plus the x , y , and z positions of the sacrum) for each of the 17 subjects across a range of running speeds. Section IV summarizes the results and examines the differences and similarities between and across subjects, groups, speeds, and legs.

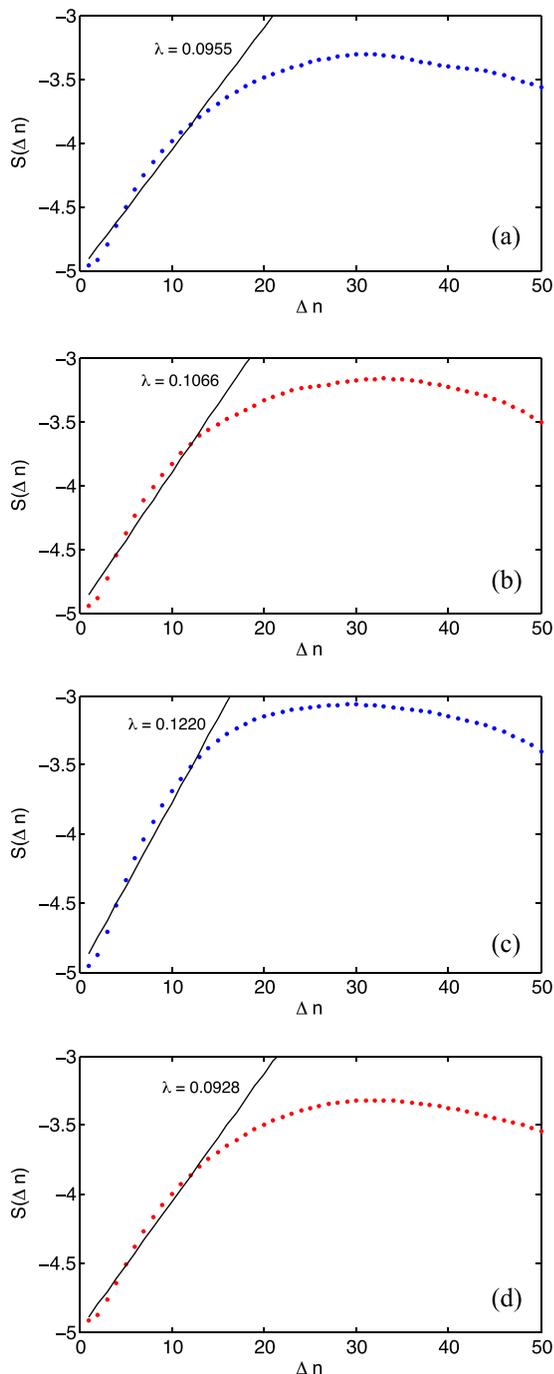


FIG. 5. Lyapunov exponent calculations for the embedded data of Figure 3. The slopes of the scaling regions of these curves—fit by the superimposed lines in the plots—represent estimates of the maximal Lyapunov exponent λ_1 of the corresponding trajectories. As before, time is plotted in units of the sampling interval $\Delta n = 1/300$ th s. The stride intervals in these trials were 0.65 and 0.71 s for the NA and WA subjects—i.e., 196 and 211 sampling intervals, respectively.

IV. RESULTS

A. Knee dynamics

Our analysis of the embedded knee-joint dynamics supports our first hypothesis: λ_1 generally increases with running speed for all runners, with and without amputations. The average λ_1 of the right and left knee angles of the NA runners increased from 0.095 and 0.101 at 3 m/s to 0.137 and

0.136 at 9 m/s, respectively. The overall trends—higher λ_1 at faster running speeds—were similar for the WA subjects, further supporting our first hypothesis. The average λ_1 of WA subjects was 0.103 and 0.090 for affected and unaffected legs, respectively, at 3 m/s; at 9 m/s, the corresponding values were 0.138 and 0.124. Not all of these increases are monotonic; this is not surprising in a study with only six WA and 11 NA subjects, given the complexity of the data-gathering task. Figure 6 plots the λ_1 values for both groups across all running speeds.

The symmetry of the system—the subject of the second hypothesis—is reflected in the similarities and differences between the curves in Figure 6. As one would expect, the left and right knee dynamics were quite similar in the NA subjects, who have two intact biological legs—although the λ_1 values did diverge somewhat at the fastest running speed. Not surprisingly, the differences between legs in the WA subjects were more pronounced and more consistent across all speeds. This result is consistent with our second hypothesis regarding inter-leg asymmetry in this group.

B. Hip dynamics

The λ_1 values for the reconstructed hip-joint dynamics were also consistent with our first two hypotheses. The average λ_1 of NA runners was 0.098 and 0.103 at 3 m/s for the right and left legs, respectively; these values increased to 0.119 and 0.116 at 9 m/s (Figure 7).

Again, the values for the right and left legs were generally similar for the NA subjects, reflecting the symmetry in their running gait. As in the case of the knee-angle results in Sec. IV A, the embedded hip-joint data provide some indications of asymmetry in the dynamics between unaffected and affected legs of the WA subjects, again supporting our second hypothesis. The average λ_1 of WA subjects was 0.113 and 0.075 at 3 m/s for the affected and unaffected legs, respectively; these values increased to 0.131 and 0.122 at 9 m/s. The convergence of the WA λ_1 values at higher

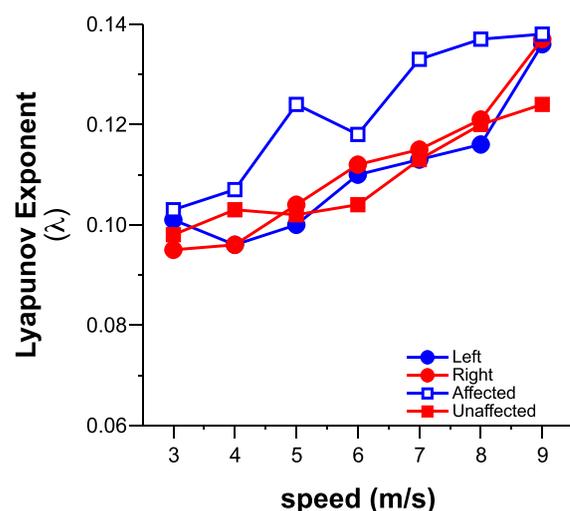


FIG. 6. λ_1 values for the embedded knee-joint dynamics of non-amputees and subjects with amputations. The values reported are in units of inverse Δn , the sampling interval of the data; to convert them to inverse seconds, one multiplies by the 300 Hz sampling rate.

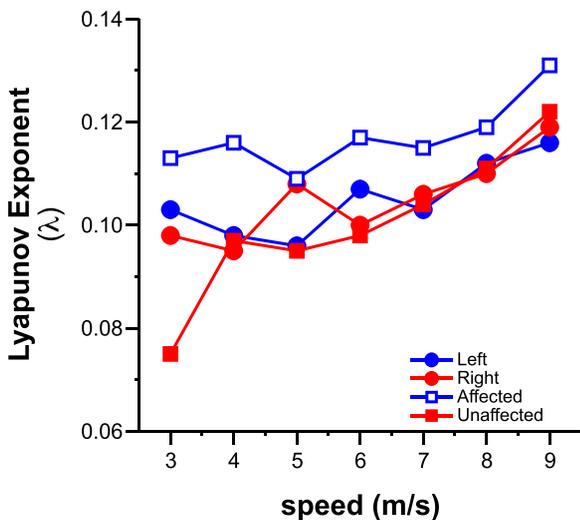


FIG. 7. λ_1 values for the embedded hip-joint dynamics of non-amputees and subjects with amputations. The values reported are in units of inverse Δn , the sampling interval of the data; to convert them to inverse seconds, one multiplies by the 300 Hz sampling rate.

running speeds that is apparent in Figure 7—a reduction in the asymmetry in the dynamics—might indicate that while there could be many different mechanical choices to run slowly (i.e., fewer constraints), there may only be one effective way to run at faster speeds.

Overall, the average λ_1 increased with running speed for both legs in both groups, again supporting our first hypothesis. Since the knee and hip angles are measurements of the same dynamical system—essentially, different measurement functions h applied to the same underlying dynamics M —these corroborations are not surprising.

C. Center-of-mass dynamics

As described at the end of Sec. II, we used the sacrum marker at the base of the spine as a proxy for the center-of-mass position. See Figure 8 for λ_1 values for the dynamics

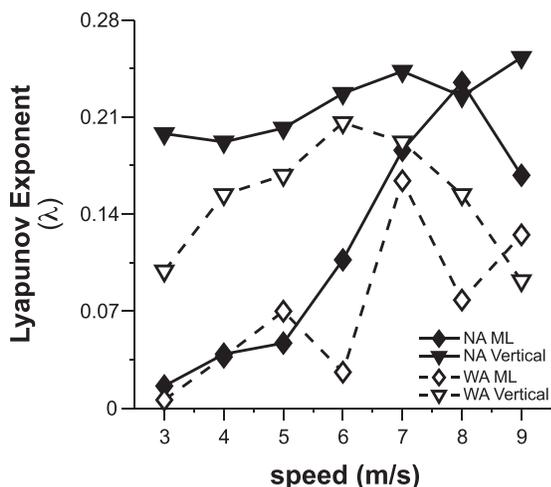


FIG. 8. λ_1 values for the embedded sacrum-position dynamics of non-amputees and subjects with amputations. The values reported are in units of inverse Δn , the sampling interval of the data; to convert them to inverse seconds, one multiplies by the 300 Hz sampling rate.

reconstructed from the medio-lateral and vertical positions of this marker. (The anterior-posterior position of the sacrum during treadmill running reflects more about the subjects’ ability to match treadmill speed than anything else, and hence was not included in these analyses.) For reasons that are discussed in Sec. V, the nonmonotonicity in these curves is more pronounced than in the hip and knee data. The λ_1 of the medio-lateral dynamics of the sacrum marker generally increased with running speed, as did the λ_1 of the vertical dynamics of the NA runners. These findings are in accordance with our first hypothesis. However, the dynamics reconstructed from time-series data of the vertical position of the sacrum in WA subjects exhibited a different pattern: first rising and then falling as running speed increased. This pattern is significantly different from those in the hip- and knee-joint dynamics. Since all of these data are simultaneous measurements of different macroscopic variables in the same dynamical system, this discrepancy between joint dynamics and center-of-mass dynamics is a puzzling finding from a dynamical-systems standpoint; see Sec. V for more discussion of this issue.

The sacrum position data also had interesting implications regarding our third hypothesis (that the center-of-mass dynamics of WA runners will be less stable than in NA runners³⁷). The answer appears not to be so simple. Across all speeds, λ_1 was smaller in the vertical dynamics for WA runners—i.e., those dynamics were more stable. In the medio-lateral direction, WA runners were as stable as NA runners at slower speeds, but the λ_1 values diverged at higher speeds.

V. DISCUSSION

A nonlinear time-series analysis of knee, hip, and sacrum dynamics of runners with (NA) and without (WA) a unilateral transtibial amputation confirmed our hypothesis that λ_1 generally increases with running speed, with one exception: the vertical position of the sacrum marker of WA runners, where the λ_1 of the embedded time-series data first rose and then fell as running speed increased. It may be the case that runners exert increased control of the core to compensate for decreased stability elsewhere, and this effect may be more pronounced in WA runners.

Our second hypothesis concerned symmetry in the embedded lower-limb dynamics for WA subjects. Our analysis indicated that the λ_1 of the embedded time-series data from the affected leg was indeed higher than for the unaffected leg—except for knee data at low speeds, where this difference was present but not pronounced.

Our third hypothesis—that the λ_1 of the center-of-mass dynamics of WA runners would be higher than for NA runners—was not verified by this analysis, except for a single midrange speed (5 m/s) in the embedded medio-lateral sacrum position data. This may be due to the effects noted in the first paragraph of this section. It is also important to note that we observed small amounts of nonstationarity in the medio-lateral data due to subtle changes of the subjects’ side-to-side positions on the treadmill. Although others have minimized these kinds of nonstationarities in the signal using divided difference methods prior to computing Lyapunov exponents,¹⁰ we avoided that approach because difference

methods can amplify any inherent noise in the signal. Moreover, we believe that these slight nonstationarities may represent behavior that is dynamically meaningful, as opposed to the kind of unavoidable drift that occurs in a measurement sensor. That is, slight changes in the subject's position from step to step may represent responses to local disturbances during running, thus providing additional insight into dynamic stability.

Readers from the biomechanics community will have noted that we did not do any of the traditional statistics analyses on these results—e.g., fitting a regression line to the data in the tables and giving an R^2 value to quantify our certainty about whether or not those data validate a particular hypothesis. Numerical algorithms that extract important properties of complicated nonlinear dynamical systems are based on approximations of the associated theory. They involve a number of parameters that strongly affect the results, and they are notoriously sensitive to noise, data length, and other sampling effects. Because of this, algorithms like `lyap_k` may inject systematic biases in the results, which violates the underlying assumptions of traditional statistics.³⁸ A systematic exploration of these effects is mandatory if one is to believe the results: minimally, a comparison of the results of different algorithms and a careful exploration of the parameter space of each one. We performed all of these kinds of checks on our results, as described in Sec. III.

At this point, we are unaware of any other studies that have quantified the nonlinear dynamics of time-series data for individuals with unilateral amputations running across a range of speeds, as described here. Our findings on this unique population of runners, then, are difficult to compare directly to other work. Enoka *et al.*¹² were the first to provide important insights into the asymmetries that exist between the biological and prosthetic leg in individuals with unilateral amputations. As was normal in that era, the runners with amputations used inelastic prostheses designed for walking, not running. Yet, they were able to run at speeds ranging from 2.7 m/s–8.2 m/s and exhibited notable kinematic intra-limb asymmetries, e.g., significant reductions in the joint angle range of motion of the prosthetic leg compared to the biological leg. The leg prostheses used by runners in our study were designed to mimic the spring-like mechanical behavior of biological legs more closely. Even so, we observed slight asymmetries in the stability of the hip and knee dynamics. We also observed slight asymmetries in the stability of the hip and knee dynamics, suggesting that running-specific prostheses do not yet exactly replicate the biomechanical function of biological legs.

Data issues—number and length of trials, as well as noise level and sampling effects—are present in any experimental study, especially one involving complex laboratory apparatus and human subjects at the edge of their normal operating regime. Our study involved only 17 subjects, not all of whom ran at all speeds. The “due diligence” demanded by nonlinear time-series analysis—careful examination of the geometry of the results produced by the algorithms, a thorough exploration of the parameter space of those algorithms, and rejection of any data that did not clearly pass those checks—further reduced the number of trials that are

averaged into the numbers reported here. This issue was strongest in the WA data, where we had only six subjects, and in the medio-lateral direction, where the subjects' sideways drift on the treadmill challenged the `lyap_k` algorithm. The 3 m/s and 8 m/s points on the WA medio-lateral λ_1 curve in Figure 8, for instance, are averages across only two trials, which makes them somewhat suspect. One of the trials that is averaged into the 9 m/s NA medio-lateral λ_1 is also problematic because its λ_1 value drops precipitously from 8 to 9 m/s. We suspect that this is due to the natural nonstationarities that are involved in reaching one's top speed. Top-speed trials often ended with a high-speed treadmill dismount. In all of our analyses, we discarded the first and last segment of the data (resp., before the subject's gait equilibrated and after s/he dismounted from the treadmill), but the dynamics may also be changing in the time period leading up to the dismount. This brings up another issue. It is well known that the effectiveness of algorithms for calculating λ_1 improves with the length of the time series. Bruijn *et al.*⁴ noted this effect in the context of human walking studies, where λ_1 values stabilized when the length of the time series reaches a few hundred strides. Walking is a very different gait, however, so one cannot extrapolate those effects to running and sprinting. And because those gaits cannot, by definition, be sustained for very long, we cannot, unfortunately, explore this effect across all of our data.

Readers from the nonlinear dynamics community will have noted the differences between the λ_1 values. This bears some explanation since the different time-series data sets studied here are simultaneous samples of the same nonlinear dynamical system. Theoretically, the λ_1 values of the dynamics reconstructed from these different time-series datasets should be the same. This holds if the sensors that measure those different angles produce smooth, generic functions of at least one state variable of that system, and as long as the dynamics themselves are smooth. In practice, the length of the datasets plays a role as well. If the dynamical coupling between parts of the body is weak, that coupling will not manifest during a short time series and thus the “invariants” of the reconstructed dynamics will not be the same from joint to joint. Since we were interested in the dynamics of gaits that could not be sustained indefinitely (viz., running at top speed), gathering longer time series was not an option. The λ_1 values reported here, then, are really more like local λ_s ,² also known as finite-time Lyapunov exponents, and they should not be expected to be identical across the entire body. There may be other effects at work here: the sharp ground-contact forces of running may disrupt the smoothness of the dynamics, and the movement of the residual limb within the prosthetic socket (“pistoning”) may add dynamics.

This study raises a variety of interesting questions regarding stability, symmetry, and the effects of a running-specific prosthesis. In this paper, we follow the practice of defining dynamic stability as the resistance to a small perturbation. Perturbation size is an obvious issue here. A large-enough perturbation may bump the trajectory completely out of the basin of attraction of the almost-periodic running gait attractor—or cause the dynamics to bifurcate altogether. This is third criterion of Bruijn *et al.*³ for gait stability: “the

largest recoverable perturbation specified by the limits of the system needs to be larger than the perturbations encountered.” If this limit is exceeded, falls may occur. Our work focused on the small-scale perturbations that occur during normal running and how they evolve over time scales smaller than the stride interval. While large-scale perturbations and fall dynamics are important, we believe that these small-scale instabilities are also dynamically and biologically meaningful.

Perturbation provenance is an even more complicated issue here. It is not clear whether the body reacts differently to endogenous perturbations (e.g., in the neuromuscular system) versus exogenous perturbations, like friction or irregularities in the running surface. The analysis presented in this paper explores the stability of running using the rate of divergence of nearby trajectories in the reconstructed state space. This provides some indication of how the system responds to small perturbations, but it does not distinguish between internal and external perturbations. If the system is autonomous, this distinction is irrelevant. However, if the dynamics are nonstationary—that is, if the forward evolution from a given point in state space depends on how (or when) the system reached that point—this distinction may be very important. One could explore this by delivering controlled perturbations to the subject on the treadmill and studying the resulting dynamics. These experiments would be challenging. There are a number of technical issues surrounding sampling movement trajectories following a perturbation, including the short time scales over which the body’s internal controller reacts and the potential hystereses and nonstationarities in that controller: e.g., a shift from feedback to a feedforward control strategy. For instance, the body could learn, over time, to prepare an appropriate response at the expected time of a perturbation. Since the dynamics of neuromuscular control systems can occupy different subsets of state space, depending on the context of the system,⁸ these kinds of controller actions could significantly impact any results that are based upon delay-coordinate embedding. Experiments that elucidated these effects, while challenging, have the potential to reveal general strategies of how the body’s internal controller deals with external perturbations and whether these responses can be captured by nonlinear time-series analysis.

With regards to dynamic symmetry, the anthropomorphic differences between the affected and unaffected legs of runners with a unilateral transtibial amputation are accompanied by slight asymmetries in stepping kinematics of running and sprinting.¹⁶ Interestingly, adding mass (~300 g) to the running-specific prosthesis helps to improve kinematic symmetry.¹⁶ Similarly, anthropomorphic and mass differences between the unaffected and affected leg may create stability asymmetries in the dynamics of runners with a unilateral amputation. Our analysis suggests that the ability to respond to small perturbations during running may be compromised in the affected leg as compared to the unaffected leg. It is important to note that the running-specific prosthesis plus the socket together weigh ~2.3 kg while the biological leg (foot and shank) weighs ~3.6 kg. The question remains as to whether adding mass to the running-specific prosthesis, as

explored in Ref. 16, would improve the dynamic symmetry between the unaffected and affected leg in runners with a unilateral amputation.

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- ³⁵See the penultimate paragraph of Sec. III for a detailed explanation of this point.
- ³⁶This parameter specifies the size of the neighborhood whose points are tracked for the calculation of the spreading factor S . Too-small values of ϵ cause numerical problems because the neighborhood contains only a few points; too-large values cause the calculation to sample the dynamics too broadly.
- ³⁷The second hypothesis is not at issue in this section, since the sacrum position data do not effectively isolate the dynamics of the individual lower limbs.
- ³⁸Indeed, Kantz and Schreiber²² quote Salman Rushdie to make this point.